CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2015 Assessment Examination

FORM VI

MATHEMATICS EXTENSION 2

Thursday 14th May 2015

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 73 boys

Examiner

MLS

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ has eccentricity: $\boxed{1}$

(A) $\dfrac{5}{4}$

(B) $\dfrac{4}{5}$

(C) $\dfrac{4}{3}$

(D) $\dfrac{3}{4}$

QUESTION TWO

The derivative of $x^2 \sin e^x$ is: $\boxed{1}$

(A) $x(-e^x \cos e^x + 2\sin e^x)$

(B) $x(-xe^x \cos e^x + 2\sin e^x)$

(C) $x(x \cos e^x + 2\sin e^x)$

(D) $x(xe^x \cos e^x + 2\sin e^x)$

Exam continues next page ...

QUESTION THREE

The primitive of $\dfrac{e^x}{1+e^{2x}}$ is: **1**

(A) $\log_e(1+e^{2x})+C$

(B) $\frac{1}{2}\log_e(1+e^{2x})+C$

(C) $\tan^{-1}e^{2x}+C$

(D) $\tan^{-1}e^{x}+C$

QUESTION FOUR

The polynomial equation $x^3+4x^2+2x-1=0$ has roots α, β and γ. The polynomial equation with roots α^2, β^2 and γ^2 is: **1**

(A) $x^3-12x^2+12x-1=0$

(B) $x^3+16x^2+4x+1=0$

(C) $x^3+20x^2-4x+1=0$

(D) $x^6+4x^4+2x^2-1=0$

QUESTION FIVE

Find $\arg(z+w)$ given that $z=i$ and $w=\dfrac{1}{\sqrt{2}}(1+i)$. **1**

(A) $\dfrac{3\pi}{8}$

(B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{3}$

(D) $\dfrac{3\pi}{4}$

Exam continues overleaf ...

QUESTION SIX

When $x^4 - kx + 1$ is divided by $x^2 + 1$ the remainder is $3x + 2$. The value of k is: **1**

(A) -1

(B) -2

(C) -3

(D) -4

QUESTION SEVEN

The equations of the asymptotes to the hyperbola $x^2 - 4y^2 = 4$ are: **1**

(A) $y = \frac{1}{4}x$ and $y = -\frac{1}{4}x$

(B) $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$

(C) $y = 2x$ and $y = -2x$

(D) $y = 4x$ and $y = -4x$

QUESTION EIGHT

An object is fired vertically upwards from the surface of the earth. The acceleration due to gravity at height x above the earth's surface is $\dfrac{-10}{(1 + \frac{x}{R})^2}$ m/s^2, where R is the radius of the earth in metres. Given that v m/s is the velocity, v^2 could be calculated by the integral: **1**

(A) $\displaystyle\int \frac{-10}{(1 + \frac{x}{R})^2}\,dx$

(B) $\displaystyle\int \frac{20}{(1 + \frac{x}{R})^2}\,dx$

(C) $\displaystyle\int \frac{-20R^2}{(R + x)^2}\,dx$

(D) $\displaystyle\int \frac{20R}{(R + x)^2}\,dx$

Exam continues next page ...

QUESTION NINE

Form a cubic equation with roots α, β and γ given that $\alpha\beta\gamma = 6$, $\alpha + \beta + \gamma = 5$ and $\alpha^2 + \beta^2 + \gamma^2 = 21$. **1**

(A) $21x^3 + 5x^2 + 11x + 11 = 0$

(B) $5x^3 + 11x^2 - 5x - 6 = 0$

(C) $x^3 - 5x^2 + 2x + 6 = 0$

(D) $x^3 - 5x^2 + 2x - 6 = 0$

QUESTION TEN

The points P, Q and R represent the complex numbers p, q and r respectively. Given that $q - p = i(r - p)$, triangle PQR is best described as: **1**

(A) Right angled isosceles with the right angle at P and $PQ = PR$.

(B) Right angled isosceles with the right angle at Q and $PQ = QR$.

(C) Right angled isosceles with the right angle at R and $RQ = RP$.

(D) Right angled isosceles with the right angle at Q and $PQ = PR$.

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the hyperbola defined by the parametric equations $x = 2\sec\theta, y = \tan\theta$.

(i) Find the coordinates of the point P defined by $\theta = \dfrac{3\pi}{4}$. **1**

(ii) Write down the Cartesian equation of the hyperbola. **1**

(b) Consider the ellipse $9x^2 + 16y^2 = 144$.

(i) Find the eccentricity of the ellipse. **1**

(ii) Find the coordinates of the foci. **1**

(iii) Find the equations of the directrices. **1**

(iv) Write down the equation of this ellipse in parametric form. **1**

(c) It is given that $1 + i$ is a zero of $P(z) = 2z^3 - 3z^2 + cz + d$, where c and d are real numbers.

(i) Explain why $1 - i$ is also a zero of $P(z)$. **1**

(ii) Factorise $P(z)$ over the real numbers. **3**

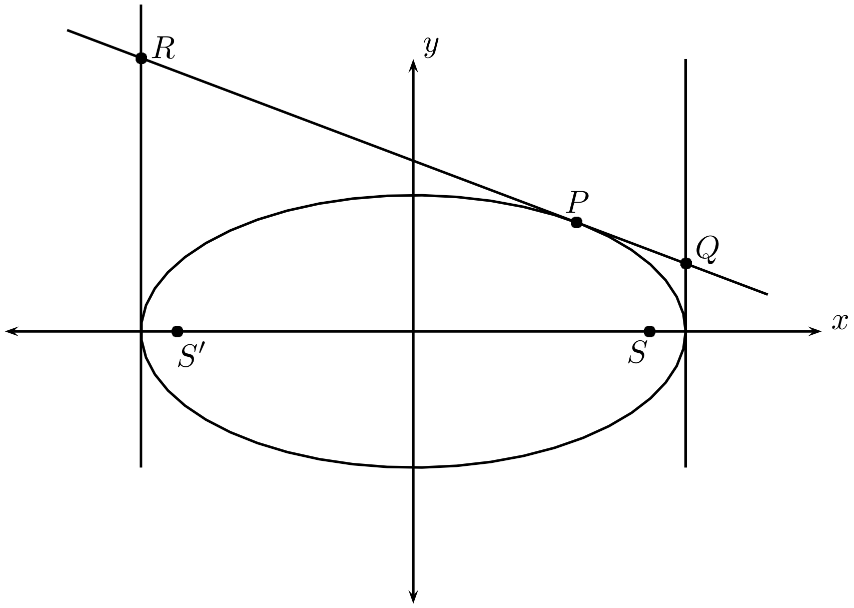
(d) The roots of $x^3 - 5x + 3 = 0$ are α, β and γ. **2**

Find a cubic polynomial with integer coefficients whose roots are $2\alpha, 2\beta$ and 2γ.

(e) An object of mass m kg moving horizontally experiences a resistive force of kv^2 Newtons, where k is a positive constant and v is the velocity of the particle in metres per second. The object starts at the origin with an initial velocity of $1\,\text{ms}^{-1}$. Find an expression for the velocity v in terms of the displacement x. **3**

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a)



The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The tangent at P meets the two vertical tangents at the points Q and R. The two foci are S and S'.

(i) Find the equation of the tangent at P. **2**

(ii) Find the coordinates of Q and R. **2**

(iii) Show that QR subtends a right angle at the focus S. **3**

(iv) Explain why QR is the diameter of the circle that passes through Q, R and S. **1**

(b) Use the substitution $t=\tan\frac{\theta}{2}$ to find $\displaystyle\int \frac{1}{1+\cos\theta-\sin\theta}\,d\theta$. **3**

(c) (i) Use the substitution $u=\frac{\pi}{2}-x$ to show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\sin x}+e^{\cos x}}\,dx.$$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx$. **2**

Exam continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that if $y = mx + k$ is a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then **3**

$$m^2a^2 - b^2 = k^2.$$

(ii) Hence find the equations of the tangents from $P(1, 3)$ to the hyperbola **3**

$$\frac{x^2}{4} - \frac{y^2}{15} = 1.$$

(b) A particle of mass m kg falls from rest in a resistive medium. The resistance to motion is of magnitude mkv when the particle has velocity v ms^{-1}. The particle reaches a terminal velocity of U ms^{-1}. Taking downwards as the positive direction, let x metres be the distance fallen in t seconds.

(i) Show that the equation of motion of the particle is $\ddot{x} = k(U - v)$. **2**

(ii) Find an expression for time t as a function of velocity v. **2**

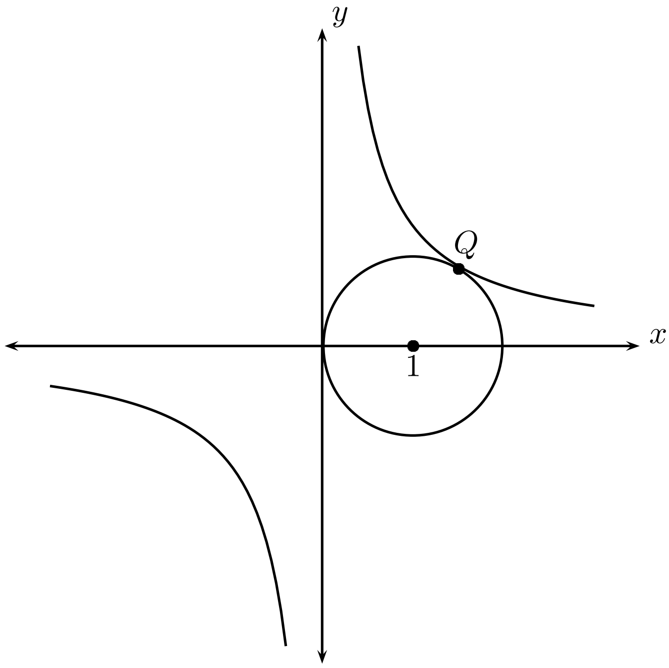
(iii) Hence find, in terms of U and k, the time T seconds taken for the particle to attain half of its terminal velocity. **1**

(iv) Find the distance fallen in this time. **3**

(v) If the particle has reached seven eighths of its terminal velocity in 15 seconds, find the value of k. **1**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



In the above diagram the hyperbola $xy = c^2$ touches the circle $(x-1)^2 + y^2 = 1$ at the point Q, but intersects it nowhere else in the plane.

(i) Prove that if the real number β is a repeated root of some polynomial equation $P(x) = 0$ then β is also a root of $P'(x) = 0$. **1**

(ii) Given that β is the x-coordinate of Q, show that β is a root of **1**

$$x^2(x-1)^2 + c^4 = x^2.$$

(iii) Show that the multiplicity of β is exactly 2. **1**

(iv) Find the value of β. **1**

(v) Explain why the other roots are complex. **1**

(vi) Hence, or otherwise, find the value of c^2 and the complex roots. **3**

Exam continues on the next page

(b) (i) Let $U_n = \int (x^2 + c)^n \, dx$. **3**

Use integration by parts to show that for constants $c > 0$ and n,

$$U_n = \frac{1}{2n+1}\Big(x(x^2+c)^n + 2nc\,U_{n-1}\Big)\,.$$

(ii) Hence, or otherwise, find $\displaystyle\int \sqrt{(x^2+c)}\,dx$. **2**

(iii) Consider the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with eccentricity e. Show that the area bounded by the axes, the right branch of the hyperbola and the line $y = a$ is **2**

$$\frac{a^3e}{2b} + \frac{ab}{4}\ln\left(\frac{e+1}{e-1}\right)\,.$$

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \ x > 0$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

MC. Form VI May Ass
Extension II

1. $b^2 = a^2(e^2-1)$

$9 = 16(e^2-1)$

$e^2 = \frac{9}{16} + 1$

$= \frac{25}{16}$

$e = \frac{5}{4}$ A.

2. $y = x^2 \sin e^x$

$\frac{dy}{dx} = x^2 e^x \cos e^x + 2x \sin e^x$

$= x(x e^x \cos e^x + 2\sin e^x)$. D.

3. $\int \frac{e^x}{1+e^{2x}} dx$

$= \int \frac{e^x}{1+(e^x)^2} dx$

$= \tan^{-1} e^x + C$ D.

4. replace x with $\sqrt{x}$

$(\sqrt{x})^3 + 4(\sqrt{x})^2 + 2\sqrt{x} - 1 = 0$

$\sqrt{x}.x + 4x + 2.\sqrt{x} - 1 = 0$

$\sqrt{x}(x+2) = 1-4x$

$x(x^2+4x+4) = 16x^2 - 8x + 1$

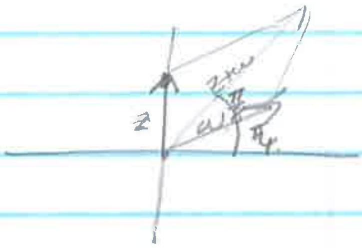
$x^3 + 4x^2 + 4x = 16x^2 - 8x + 1$

$x^3 - 12x^2 + 12x - 1 = 0$ A

Q5.

$z = i = \text{cis}\,\frac{\pi}{2}$

$w = \frac{1}{\sqrt{2}}(1+i) = \text{cis}\,\frac{\pi}{4}$



$\arg(z+w) = \frac{\pi}{8} + \frac{\pi}{4}$.

$= \frac{3\pi}{8}$ A.

Q6.

$$\begin{array}{r} x^2 - 1 \\ x^2+1 \,\big)\, x^4 + 0x^3 + 0x^2 - bx + 1 \\ x^4 \qquad\quad + x^2 \qquad\qquad \\ \hline -x^2 - bx + 1 \\ -x^2 \qquad -1 \\ \hline -bx + 2 \end{array}$$

So $-bx + 2 = 3x + 2$

So $b = -3$ C.

Q7. $y = \pm \frac{1}{2}x$ B.

Q8. $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{-10}{\left(1+\frac{x}{2}\right)^2}$

$\frac{1}{2}v^2 = -10\int \frac{1}{\left(1+\frac{x}{2}\right)^2}\,dx$.

$v^2 = \int \frac{-20}{\left(1+\frac{x}{2}\right)^2}\,dx$

$$v^2 = \int \frac{-20R^2}{(R+x)^2}\,dx \qquad C.$$

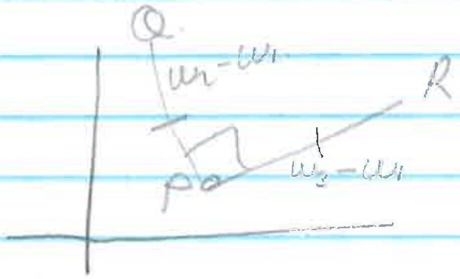
Q9. $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$25 = 21 + 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\alpha\beta+\alpha\gamma+\beta\gamma = 2.$

$x^3 - 5x^2 + 2x - 6 = 0 \qquad D.$

Q10.



A.

Q11

(a) $x = 2\sec\theta, \quad y = \tan\theta$

(i) $x = 2\sec\frac{3\pi}{4}$

$= \frac{2}{\cos\frac{3\pi}{4}}$

$= \frac{2}{-\cos\frac{\pi}{4}}$

$= -2\sqrt{2}$

$y = \tan\frac{3\pi}{4}$

$= -\tan\frac{\pi}{4}$

$= -1.$

P is $(-2\sqrt{2}, -1)$ ✓

(ii) $x^2 = 4\sec^2\theta$ $\qquad y^2 = \tan^2\theta$

$\frac{x^2}{4} = \sec^2\theta$

$(1+\tan^2\theta = \sec^2\theta)$

$\sec^2\theta - \tan^2\theta = 1$

$\frac{x^2}{4} - y^2 = 1$ ✓

(b) $\frac{9x^2}{144} + \frac{16y^2}{144} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$

(i) $b^2 = a^2(1-e^2)$ $\qquad a = 4$

$9 = 16(1-e^2)$ $\qquad b = 3$

$e^2 = 1 - \frac{9}{16}$

$= \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$ ✓

(ii) $ae = \sqrt{7}$.

Foci are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$

iii) $\frac{a}{e} = \frac{16}{\sqrt{7}} = \frac{16\sqrt{7}}{7}$.

Directrices are $x = \frac{16}{\sqrt{7}}$ or $\frac{16\sqrt{7}}{7}$

and $x = -\frac{16}{\sqrt{7}}$ or $\frac{-16\sqrt{7}}{7}$.

iv) $x = 4\cos\theta$

$y = 3\sin\theta$

c). (i) The coefficients are real so the complex zeroes occur in conjugate pairs, $\overline{z+i} = z-i$

(ii)

$$P(z) = 2z^3 - 3z^2 + cz + d$$
$$= (z-(1+i))(z-(1-i))(az+b)$$
$$= (z^2 - 2z + 2)(az+b)$$
$$= az^3 - (2a+b)z^2 + (2a-2b)z + 2b$$
$$= 2z^3 - 3z^2 + cz + d.$$

so $a = 2$, $2a - b = 3$

$b = 1$

$c = 2a - 2b$ and $d = 2b$

$= 2$ $\qquad = 2$

$$P(z) = (z^2 - 2z + 2)(2z + 1)$$

OR c(ii) $\alpha = 1+i \quad \beta = 1-i \quad \gamma$ are roots.

$$\alpha+\beta+\gamma = \frac{3}{2} \Rightarrow \quad 2+\gamma = \frac{3}{2}.$$

$$\text{so } \gamma = -\tfrac{1}{2}.$$

$$P(z) = 2(z-(1+i))(z-(1-i))(z-(-\tfrac{1}{2})).$$

$$= (z^2-2z+2)(2z+1)$$

d. $\quad x^3 - 5x + 3 \qquad \alpha, \beta, \gamma.$

Method 1 $\quad$ let $x = \frac{m}{2}$.

$$x^3 - 5x + 3 = \left(\frac{m}{2}\right)^2 - 5\frac{m}{2} + 3$$

$$= \frac{m^2}{8} - \frac{5m}{2} + 3.$$

giving $\quad \frac{x^2}{8} - \frac{5x}{2} + 3$

or $\quad x^3 - 20x + 24 = 0$

Method 2

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -5 \\ \alpha\beta\gamma &= -3 \end{aligned}\right\}$$

let new polynomial be
$$x^3 + bx^2 + cx + d$$

$$-b = 2\alpha + 2\beta + 2\gamma$$
$$= 0$$

$$c = (2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma)$$
$$= 4(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= -20$$

.... etc.

e) $m\ddot{x} = -kv^2$

$v \frac{dv}{dx} = -\frac{k}{m} v^2$ ✓

$\frac{dv}{dx} = -\frac{k}{m} v$

$\frac{dx}{dv} = -\frac{m}{k} \frac{1}{v}$

$x = -\frac{m}{k} \int \frac{1}{v} \, dv$

$= -\frac{m}{k} \ln v + c$

$x=0, v=1,$ $0 = 0 + c \Rightarrow c = 0$ need to show $c=0$

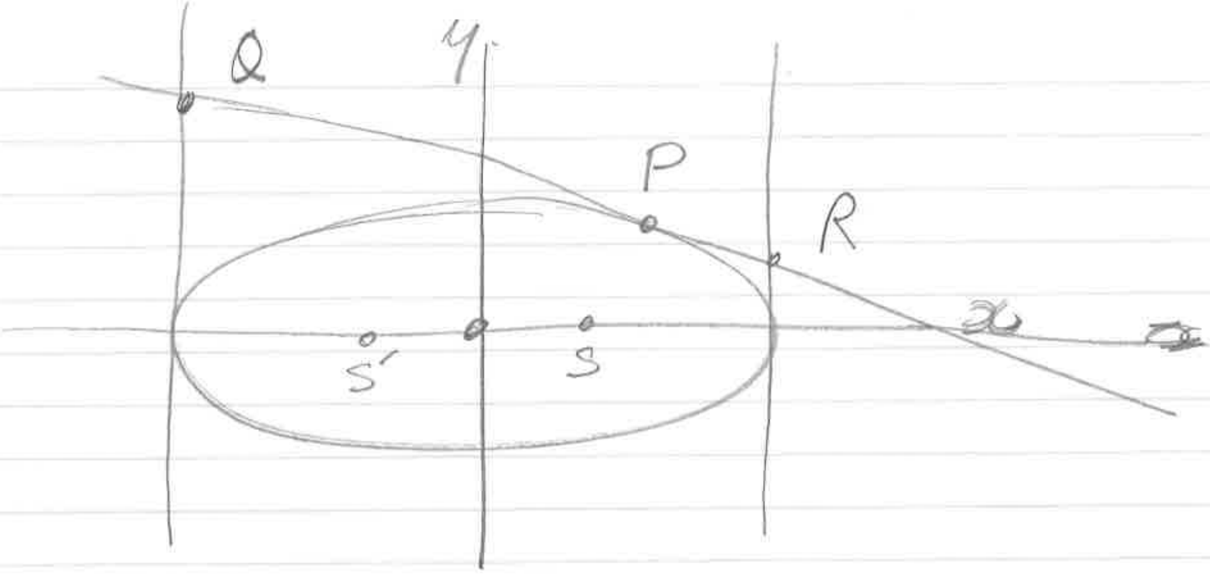
so $x = -\frac{m}{k} \ln v$ ✓

so $\ln v = \frac{kx}{-m}$

$v = e^{\frac{-kx}{m}}$ ✓

Q12.

(a)



$P(a\cos\theta, b\sin\theta)$ $\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) gradient $\qquad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}.$$

$$= -\frac{b^2}{a^2}\frac{x}{y}.$$

at P, $\qquad m = -\frac{b^2}{a^2}\frac{a\cos\theta}{b\sin\theta}$

$$= -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$$

tangent at P

$$y - b\sin\theta = \frac{-b}{a}\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)$$

$$y\sin\theta - b\sin^2\theta = -\frac{b}{a}\cos\theta\, x + b\cos^2\theta.$$

$$y\sin\theta + \frac{b}{a}\cos\theta\, x = b(\sin^2\theta + \cos^2\theta)$$

$$\frac{\sin\theta}{b}y + \frac{\cos\theta}{a}x = 1$$

ii) At Q, $x=a$

$$\cos\theta + \frac{y\sin\theta}{b} = 1$$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

At R, $x=-a$.

$$\frac{y\sin\theta}{b} - \cos\theta = 1$$

$$y = \frac{b(1+\cos\theta)}{\sin\theta}$$

✓ for at $Q, x=a$ / $R, x=-a$

✓ for y coord.

S' is $(-ae, 0)$
R is $\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$

S is $(ae, 0)$
Q is $\left(+a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$

iii)

$$m_{QS} = \frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a-ae} = \frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

✓

$$m_{RS} = \frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a-ae} = \frac{b(1+\cos\theta)}{a(1+e)\sin\theta}.$$

$$m_{RS} \times m_{QS} = \frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a(1-e)} \times \frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a(1+e)}$$ ✓

$$= \frac{b^2}{-a^2(1-e^2)}$$

now $b^2 = a^2(1-e^2)$.

$= -1$ ✓ needs $b^2 = a^2(1-e^2)$.

So $RS \perp QS$ as required

iv) The angle in a semicircle is a right angle, RQ is subtending a right angle at S. ✓

v).   2.

R

b)

$$\int \frac{1}{1-\cos\theta-\sin\theta}\, d\theta$$

$t = \tan\frac{\theta}{2}$ $\quad a=\frac{1}{2}$

$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta$

$= \frac{1}{2}(1+t^2)\, d\theta$

$\frac{2dt}{1+t^2} = d\theta$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}} \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{1+t^2+(1-t^2)-2t}$$

$$= \int \frac{2dt}{2-2t}$$

$$= \int \frac{1}{1-t}\, dt$$

$$= -\ln(1-t) + C$$

$$= -\ln\left(1-\tan\frac{\theta}{2}\right) + C$$

c)

i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx$

$u = \frac{\pi}{2} - x$
$du = -dx$

x	$\frac{\pi}{2}$	0
u	0	$\frac{\pi}{2}$

$x = \frac{\pi}{2} - u$
$\sin x = \cos u$
$\cos x = \sin u.$

$= \displaystyle\int_{\frac{\pi}{2}}^{0} \frac{e^{\cos u}}{e^{\cos u}+e^{\sin u}}\,-du.$ ✓

$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\cos u}}{e^{\cos u}+e^{\sin u}}\,du$ ✓

$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}\,dx,$ since we have a definite integral

ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}} + \frac{e^{\cos x}}{e^{\sin x}+e^{\cos x}}\,dx$

$= \frac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}+e^{\cos x}}{e^{\sin x}+e^{\cos x}}\,dx.$ ✓

$= \frac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} 1\,dx$

$= \frac{\pi}{4}$ ✓

13

(a) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $y = mx + k$.

$\frac{x^2}{a^2} - \frac{(mx+k)^2}{b^2} = 1$ ✓ at the pt of intersection

$\frac{x^2}{a^2} - \frac{m^2x^2 + 2mxk + k^2}{b^2} = 1$

$x^2a^2 - a^2(m^2x^2 + 2mxk + k^2) = a^2b^2$

$x^2a^2 - a^2m^2x^2 - 2a^2mxk - a^2k^2 = a^2b^2$

$x^2(b^2 - a^2m^2) - 2a^2mxk - a^2(k^2 + b^2) = 0.$

Tangent so $\Delta = 0$

$(2a^2mk)^2 - 4(b^2 - a^2m^2)(-a^2k^2 - a^2b^2) = 0$ ✓

$4a^4m^2k^2 - 4(-b^2a^2k^2 - a^2b^4 + a^4k^2m^2 +$

$a^4m^2b^2) = 0$ ✓ for tidying up without cheating

$4b^2a^2k^2 = -4a^2b^4 + 4a^4m^2b^2$

$k^2 = a^2m^2 - b^2$

award ✓ if they do nothing correct but get $m = \frac{b^2x_1}{a^2y_1}$

(ii) tangent is $y = mx + k$

$k = y - mx$

at $(1,3)$, $k = 3 - m$ ✓ for using $(1,3)$ in $y = mx + k$

So $k^2 = 9 - 6m + m^2$

But $k^2 = a^2m^2 - b^2$ from (i)

so $k^2 = 4m^2 - 15$

$9 - 6m + m^2 = 4m^2 - 15$

$3m^2 + 6m - 24 = 0$

$m^2 + 2m - 8 = 0$

$(m+4)(m-2) = 0$

$m = -4$ or 2

$m = -4 \Rightarrow k = 7$

tangent is $y = -4x + 7$

$m = 2 \Rightarrow k = 1$

tangent is $y = 2x + 1$

Q13.

b (i)

$t=0$
$x=0$
$v=0$.

$$m\ddot{x} = mg - mkv.$$
$$\ddot{x} = g - kv$$

as $\ddot{x} \to 0$, $v \to \frac{g}{k}$ or $\ddot{x}=0$, $v=U$

So $U = \frac{g}{k}$

$$\ddot{x} = k\left(\frac{g}{k} - v\right)$$
$$= k(U-v)$$

ii) Time $\frac{dv}{dt} = k(U-v)$

$$\frac{dt}{dv} = \frac{1}{k}(U-v)$$

$$\int_0^t dt = -\frac{1}{k}\int_0^v \frac{-1}{U-v}\,dv$$

$$t = -\frac{1}{k}\left[\ln(U-v)\right]_0^v$$
$$= -\frac{1}{k}\left[\ln|U-v| - \ln|U|\right]$$
$$= \frac{1}{k}\ln\left|\frac{U}{U-v}\right|$$

(iii) $v = \frac{1}{2}U$, $t = T = \frac{\ln 2}{k}$

(iv) $v\frac{dv}{dx} = k(U-v)$

$$\frac{dx}{dv} = \frac{1}{k}\,\frac{v}{U-v}.$$

$$\int_0^x dx = \frac{-1}{k}\int_0^v \frac{-v}{U-v}\,dv$$ ✓

$$x = -\frac{1}{k}\int_0^v \frac{U-v}{U-v} - \frac{U}{U-v}\,dv.$$

$$= -\frac{1}{k}\int_0^v 1 + U\frac{-1}{U-v}\,dv$$ ✓

$$= -\frac{1}{k}\Big[v + U\ln|U-v|\Big]_0^v$$

$$= -\frac{1}{k}\big[v + U\ln|U-v| - (0 + U\ln U)\big]$$

$$= -\frac{1}{k}\Big[v + U\ln\Big|\frac{U-v}{U}\Big|\Big]$$

when $v = \frac{U}{2}$

$$x = -\frac{1}{k}\Big[\frac{U}{2} + U\ln\Big(\frac{\frac{1}{2}U}{U}\Big)\Big]$$

$$= -\frac{1}{k}\Big(\frac{U}{2} + U\ln\frac{1}{2}\Big)$$ $\ln\frac{1}{2} = -\ln 2.$

$$= \frac{U}{k}\Big(\ln 2 - \frac{1}{2}\Big).$$ ✓

v) when $t = 15$, $v = \frac{7}{8}U$

$$t = \frac{1}{k}\ln\Big|\frac{U}{U-\frac{7}{8}U}\Big|$$

$$15 = \frac{1}{k}\ln 8$$

$$k = \frac{\ln 8}{15}$$ ✓

$$= \frac{3\ln 2}{15} = \frac{1}{5}\ln 2.$$

Q14.

a(i) if β is a repeated root of $P(x)=0$, then $(x-\beta)$ is a factor of multiplicity at least two, i.e. $(x-\beta)^2$ is a factor of $P(x)$.

Let $P(x) = (x-\beta)^2 Q(x)$ for some polynomial $Q(x)$.

Then
$$P'(x) = 2(x-\beta)Q(x) + (x-\beta)^2 Q'(x)$$
$$= (x-\beta)\left[2Q(x) + (x-\beta)Q'(x)\right]$$

So $x-\beta$ is a factor of $P'(x)$.

Thus
$$P'(\beta) = (\beta-\beta)\{2Q(\beta) + (\beta-\beta)Q'(\beta)\}$$
$$= 0$$

i.e. β is a root of $P'(x)=0$.

(ii) To find Q, we need to intersect
$$(x-1)^2 + y^2 = 1 \qquad (1)$$
and
$$xy = c^2 \qquad (2)$$

$x^2 \times (1)$ gives
$$x^2(x-1)^2 + x^2y^2 = x^2$$
but from (2)
$$x^2y^2 = c^4$$
hence
$$x^2(x-1)^2 + c^4 = x^2$$
as required.

(iii) let
$$P(x) = x^2(x-1)^2 + c^4 - x^2$$
$$= x^2(x^2-2x+1) + c^4 - x^2$$
$$= x^4 - 2x^3 + c^4$$
$$P'(x) = 4x^3 - 6x^2$$
$$P''(x) = 12x^2 - 12x$$
$$= 12x(x-1)$$

$P''(x)=0$ has roots $x=0, 1$

By inspection, neither is a root of $P(x)$ as $c^2>0$

Hence $P(x)$ has <u>no</u> roots of multiplicity 3 or more.

But it does have a root of multiplicity 2, since the hyperbola is tangent to the circle at Q (so the multiplicity of β is at least 2)

(iv) β must be a root of $P'(x)=0$ AND $P(x)=0$.

roots of $P'(x)=0$ are given by

$$2x^2(2x-3)=0$$

$$x=0 \quad , x=\frac{3}{2}$$

but $x=0$ is <u>not</u> a root of $P(x)=0$.

Hence $\beta=\frac{3}{2}$.

(v) $P(x)$ is degree 4, hence it has exactly 4 roots to $P(x)=0$ (possibly repeated). Since the root $x=\beta$ is a double root, there must be two more roots. Since the graphs do not intersect anywhere else in the real plane (given), the roots must be complex. (indeed, they must be complex conjugates).

(vi) Since $P(\beta)=0$,

$$x^4-2x^3+c^4=0 \quad \text{when } x=\frac{3}{2}$$

$$\left(\frac{3}{2}\right)^4-2\left(\frac{3}{2}\right)^3+c^4=0$$

$$c^4=\frac{27}{8}\left(2-\frac{3}{2}\right)$$

$$=\frac{27}{16}$$

$$c^2=+\sqrt{\frac{27}{16}}$$

$$=\frac{3\sqrt{3}}{4}$$

let the roots of $P(x)=0$ be $\frac{3}{2}, \frac{3}{2}, \alpha, \bar{\alpha}$

Then

$$\frac{3}{2}+\frac{3}{2}+\alpha+\bar{\alpha} = 2 \quad (\text{sum of the roots})$$

$$\alpha+\bar{\alpha} = -1$$

$$2\,\mathrm{Re}(\alpha) = -1$$

$$\mathrm{Re}(\alpha) = -\frac{1}{2}$$

also

$$\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\alpha\bar{\alpha} = \frac{27}{16} \quad (\text{product of roots})$$

$$\alpha\bar{\alpha} = \frac{3}{4}$$

note that $\alpha\bar{\alpha} = |\alpha|^2$, so

$$\mathrm{Re}(\alpha)^2 + \mathrm{Im}(\alpha)^2 = \frac{3}{4}$$

$$\frac{1}{4} + \mathrm{Im}(\alpha)^2 = \frac{3}{4}$$

$$\mathrm{Im}(\alpha)^2 = \frac{1}{2}$$

$$\mathrm{Im}(\alpha) = \pm\frac{1}{\sqrt{2}}$$

the roots are $\frac{3}{2}, \frac{3}{2}, -\frac{1}{2} \pm \frac{1}{\sqrt{2}}i$

b(i) $$U_n = \int (x^2+c)^n\,dx$$

$$= \int 1 \times (x^2+c)^n\,dx$$

$$= x \times (x^2+c)^n - \int x \times 2xn\,(x^2+c)^{n-1}\,dx$$

$$= x(x^2+c)^n - 2n\int x^2(x^2+c)^{n-1}\,dx$$

$$= x(x^2+c)^n - 2n\int (x^2+c-c)(x^2+c)^{n-1}\,dx$$

$$= x(x^2+c)^n - 2n\left\{\int (x^2+c)(x^2+c)^{n-1}\,dx - c\int (x^2+c)^{n-1}\,dx\right\}$$

$$= x(x^2+c)^n - 2n(U_n - cU_{n-1})$$

so

$$U_n = x(x^2+c)^n - 2nU_n + 2ncU_{n-1}$$

$$U_n(1+2n) = x(x^2+c)^n + 2ncU_{n-1}$$

$$U_n = \frac{1}{2n+1}\left\{x(x^2+c)^n + 2ncU_{n-1}\right\}$$

(ii) let $n=\frac{1}{2}$. Then

$$U_{\frac{1}{2}} = \int (x^2+c)^{1/2}\,dx$$

$$= \frac{1}{1+1}\left(x(x^2+c)^{1/2} + cU_{-\frac{1}{2}}\right)$$

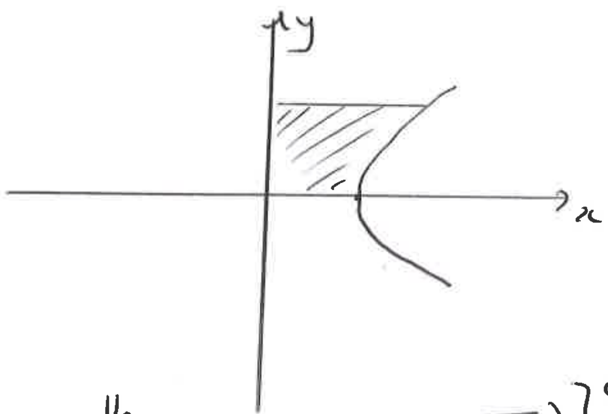
$$= \frac{1}{2}x(x^2+c)^{1/2} + \frac{1}{2}c\int \frac{1}{\sqrt{x^2+c}}\,dx$$

$$= \frac{1}{2}x(x^2+c)^{1/2} + \frac{1}{2}c\ln\left(x+\sqrt{x^2+c}\right)$$ (from the supplied table of integrals)

(iii) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

hence $$x^2 = a^2\left(1+\frac{y^2}{b^2}\right)$$

$$= \frac{a^2}{b^2}(y^2+b^2)$$



The required area is

$$\frac{a}{b}\int_0^a \sqrt{y^2+b^2}\,dy = \frac{a}{b}\left[\frac{1}{2}y(y^2+b^2)^{1/2} + \frac{1}{2}b^2\ln\left(y+\sqrt{y^2+b^2}\right)\right]_0^a$$

$$= \frac{a}{2b}\left[a(a^2+b^2)^{1/2} + \frac{1}{2}b^2\left(\ln\left(a+\sqrt{a^2+b^2}\right) - \ln b\right)\right]$$

but $b^2 = a^2(e^2-1)$, so $a^2+b^2 = a^2e^2$;

$$\text{Area} = \frac{a^2}{2b}(ae) + \frac{1}{2}ab\ln\left(\frac{a+ae}{b}\right)$$

$$= \frac{a^3e}{2b} + \frac{1}{4}ab\ln\left(\frac{a^2(1+e)^2}{b^2}\right)$$

$$= \frac{a^3e}{2b} + \frac{1}{4}ab\ln\left(\frac{(1+e)^2}{e^2-1}\right)$$

$$= \frac{a^3 e}{2b} + \frac{1}{4} ab \ln \frac{(1+e)^2}{(e-1)(e+1)}$$

$$= \frac{a^3 e}{2b} + \frac{1}{4} ab \ln \frac{1+e}{e-1}$$

as required.